# Functions at the transition between French upper secondary school and University

## Communication of the CI2U

**Abstract.** A questionnaire was proposed in more than seven universities in France to detect students' skills in the fields of sequences, functions and general reasoning. The communication shows some results in the domain of functions. We introduce four levels of students' conceptions of a function at the beginning of the university: punctual, global, local and subglobal level. The results of the questionnaire confirm some older works about the teaching at the secondary school which has more or less banished the local level and has also contributed the dissociation of punctual and global levels. The analysis of university exercises sheets and curricula also stresses that university teaching has not taken sufficient responsibility for the transition towards the expected local level.

## Introduction

The CI2U (Commission Inter-Irem Université[1]) brings together teachers at university level and teachers of secondary schools interested in problems of teaching at the beginning of the university. During the last year, the commission has investigated knowledge of pupils coming from secondary schools and entering university. A questionnaire was proposed in more than seven universities in France to detect students' skills in the fields of sequences, functions, and general reasoning. The communication aims to show some results in the field of functions. This topic has been investigated by many studies over the last few decades. With this contribution, we would like to illustrate that didactical knowledge about student's conceptions of functions at the end of secondary school gained in previous studies is still valid and that the gap remains between these conceptions, the associated skills and the mathematical activity expected at university level.

The aim of this paper is to revisit didactic knowledge about functions at this school level and initiate discussions in the TSG5 about all the associated problems. We first recall some well known points regarding the transition between secondary school and university. Then we summarize some didactic knowledge about functions by introducing four levels of conceptions of the notion. Then we present the questionnaire, focusing on questions about functions and we give the results of our analysis according to the four levels of conceptions. Finally some examples of exercises given during sessions at the start of university courses, showing the gap between students skills and expected abilities are presented.

### 1) A few words about the transition between upper secondary school and university

Many studies have already characterized the specific nature of the transition between secondary school and university. For instance, Robert (1998) noticed a higher level of conceptualisation of the knowledge used at university level. This is close to the distinction made in APOS theory (Dubinsky, 1991) between Process and Object level of a given mathematical notion. We will develop this idea for the notion of function, introducing four levels of conception existing from secondary to university teaching. Sfard (1991) also claims

[1] University Inter IREM Commission

that there is a deep gap between operational and structural conceptions. We will see that the concept of function can be defined both structurally and operationally.

Robert also noted a distribution between the types of mathematical notions which is different from secondary school to university, especially the emergence at university level of new mathematical notions carrying a high level of formalism and generalisation. Finally, she pointed out the differences in the level and the nature of tasks (necessity of available knowledge, necessity of flexibility in this knowledge, for instance use of different settings and representation systems, new requirements in term of proofs at the university level…). On this point, Bloch (2004) made a step forward by clarifying the new requirements at the beginning of university by introducing nine didactic variables whose values represent factors of rupture between university and upper secondary school.

**2) The complex notion of function**

The notion of function is at the intersection of several mathematical fields (real numbers, limits, algebra, etc…) and requires the consideration of several representation systems (graphical, symbolic, algebraic etc…). Functions are complex objects which are still being learnt when students enter university. However, observations of our university colleagues show that average students trace graphs only when asked explicitly to do so; they do not think of graphs spontaneously. Moreover, average students are unable to exploit a function which is not given in an algebraic form. Students can not move from one representation system to another and the current practice of teaching in secondary schools seems to reinforce the idea that a function is usable only if its algebraic form is known.

To structure the discussion in TSG5 about this complex notion of function and these observations, we propose to introduce the following four **levels of students' conceptions** of a function when beginning university: the punctual level, the global level, the local level and the subglobal level. Punctual, global and subglobal levels match more or less the process, object and schema levels of APOS theory. However, the local level is added in our proposal and it seems to be an important one for studying the notion of function at undergraduate level. We explain these levels:

1) The first level where a function is considered as a correspondence "law", is called the **PUNCTUAL level**: a function expresses the correspondence between two sets of real numbers, an element of the first set being associated with a unique element of the second set. This level is in accordance with the usual definition of a function given in textbooks at grade 10, three years before the beginning of university. At this level, functions are represented by arithmetic formulas, that operate as a program for a calculation such as calculus programming. A table of values is then a good representation of a function, especially for pupils who consider only integers on the real line. This view of the beginning of calculus in secondary school can be considered as a first level of conceptualisation where for instance 0,9999... is the number just before 1 in students' conceptions.

2) The second level where functions are considered as objects in themselves, is called the **GLOBAL level**: this level is mainly exploited, during grade 10 to 12, because the functions

met by the pupils are almost always continuous or derivable overall. It is the level of work on algebraic expressions which can be derived for the study of the global variations of the functions. However, as pointed out by Coppé and al (2007), the table of variation of a function is a good representation of the function at this level while the algebraic expression or the graphical representation of the function can exploit and be exploited at the punctual level as well as at the global level: for instance the graph of a function is drawn point by point, but it must be considered then as a global object representing this function. This global level of conception of a function may belongs to a second level of conceptualisation for the beginning of analysis where R is seen as a whole.

3) The third level we introduce is the **LOCAL level**: limit is the basic notion of calculus and it is well known that its control requires a number of obstacles to be overcomed, about the structure of the real line for instance or the notion of equality between two real numbers. These obstacles are characteristic of this third level of conceptualisation at the beginning of calculus and working at university level on functions implies that students reach this third level of conceptualisation. This third level of conceptualisation also brings about a high level of formalisation which accentuates difficulties for students. This level is necessary as soon as one wants to approach equivalents of functions, Taylor expansions near some points and so on.

4) The fourth level is the **SUBGLOBAL** level where functions belong to function sets, such as the set of continuous functions or derivable functions and this level of reasoning is necessary as soon as one wants to study the structure of all these sets (inside a fourth level of conceptualisation).

Some older works have stressed that teaching at secondary school has more or less banished the local level and has also contributed to the dissociation of punctual and global levels. As Comin (2005) announces, "the epistemological analysis of the concept of function and variable leads us to pose that it is the idea of dependency between magnitudes which grounds the concepts of function and variable (…) The "ensemblist" approach of the concept by a correspondence between two sets modelled by a graph evacuates this idea of constraint between two magnitudes. The activity which is proposed to pupils at secondary school relates to a finite number of values and moves the pupils away from the idea of variability and continuity". There is, moreover, at secondary school, a large algebraisation at the global level which is juxtaposed for some pupils with the punctual level and these pupils can not move from one level to another. The function is seen either as a punctual process or like a global object according to the situation and without a true articulation between the two views.

Curricula have also given great attention to the study of some stereotype functions ("fonctions de références") since 1985, allowing an strong algebraisation of the calculation of the limits by comparison with these stereotypes. The part of reasoning devoted to activities at the local level has been undervalued and at the same time the work on tables of values and graphical representations has been raised. After 1990, the work on stereotype functions was slightly diminished but the activity at the local level did not appear. The graphical representation gained a larger role in the curriculum to illustrate some analytical proposals whose proofs are not assumed by the teaching at this level. According to Bloch (2002) « cette illustration des propriétés est supposée s'appuyer sur l'intuition graphique. Elle ne questionne pas le rapport graphique / fonctions supposé transparent : les élèves sont supposés voir dans le dessin graphique ce qu'y voit le professeur[2] ». After 2000, the algebraic representation system was no longer dominant and there was an attempt to rebalance the representation systems. However, as noticed by Coppé and al (2007), the algebraic register for studying functions remains the most usual register (from 30% to 58% of exercises according to a textbook at grade 10).

Coppé et al also show that pupils have more difficulty using the symbolic register of the table of variations, which belongs to the global level of functions, than in using numerical tables of values, which belong to the punctual level. In the same way, a conversion from a table of variation to another representation system (algebraic, graphical and even numerical) seems to be more difficult than a conversion from a table of numerical values. Then, to be able to move from one semiotic register to another, as noticed by Duval (1993), is not sufficient to ensure the learning of functions because registers are not equivalent with regard to punctual and global levels. Bloch (2003) points out registers which are reductive or productive depending on the punctual and the global levels and she also notes the poor work at local level. For instance, she stresses that the graphical register is produced at both levels but pupils rarely consider its power at the global level. She puts forward proposals for a new teaching of functions, supported by the global power of the graphical register. Tall (2006) introduces the notion of *procept* to qualify objects such as algebraic formulas or graphical representations which can be manipulated by students as processes as well as objects, being a possible bridge between the two levels of conceptions.

At the end of secondary school, Coppé and al (2007) note an improvement in the pupils' realisation of conversions but the concept of functions is no longer explicitly at stake for learning. There is still an algebraisation of certains techniques for studying functions, introducing global rules for computations of limits involving exponentials, polynomials and logarithms, generalizing algebraic rules for computing derivatives and finally reinforcing pupils' abilities at the global level without any strong articulations with the punctual one. Pupils can be asked about the rate of change of some functions explicitly, but the idea of the computation is not asked to be available for pupils. Moreover, it is always the same algebraic technique of computation which gives the non problematic expected result.

**3) Our questionnaire and the levels of conceptions of students' notion of function**

As recalled at the beginning of this paper, the CI2U has tried to investigate knowledge of pupils coming from secondary school and entering the university, in the first year of a science degree. A questionnaire was proposed in seven universities in France to detect students' skills in fields of sequences, functions, and general reasoning. See below.

[2] This illustration of functions' properties is supposed to be based on graphical intuition. The balance between graphic and algebraic expression seems to be transparency : students are supposed to see on the drawing what the teacher sees.

1. Parmi ces suites données par leur terme général, quelles sont celles qui ont une limite quand *n* tend vers l'infini ? Précisez cette limite quand elle existe.

- 1-1 : ______________________________
- 1-2 : ______________________________
- 1-3 **sin**(2π***n***) :

  ______________________________
- 1-4 **cos()** **:**

  ______________________________

2. Donnez les limites des fonctions suivantes :

- . 2-1a Limite quand *x* tend vers plus l'infini :

  ____________________

  2-1b Limite quand *x* tend vers en 0 :

  ____________________

  2-1c Limite quand *x* tend vers moins l'infini :

  ____________________
- . 2-2a Limite quand *x* tend vers plus l'infini :

  ____________________

  2-2b Limite quand *x* tend vers 0 :

  ____________________

  2-2c Limite quand *x* tend vers moins l'infini :

  ____________________
- ***j(x)*= cos(2π*x*)** . 2-3 Limite quand *x* tend vers plus l'infini :

  ____________________
- . 2-4a Limite quand *x* tend vers plus l'infini : ____________________

  2-4b Limite quand *x* tend vers 2 :

Table 1: questions concerning functions and sequences in the questionnaire

As regards functions and sequences, we would like to show if results given in the previous section remain valid by asking questions about limits. Functions and sequences have been chosen according to the importance of punctual and global conceptions to guess their limits. That is to say, having only a punctual or a global conception can be a help or a hindrance in guessing the right answer : that is especially the case for lim **sin**(2π***n***) when n goes to infinity and lim **cos(2π*x*)** when x goes to infinity. We have also added functions and sequences where algebraic manipulations can lead to the answer (; ;) and a last question where the local conception of functions has to be available to answer : when x goes to 2.

298 answers were analysed and the distribution of these answers among right and wrong ones was the following :

| 1-1 | 1-2 | 1-3 | 1-4 | **2-1.a** | **2-1b** | **2-1c** | **2-2a** | **2-2b** | **2-2c** | 2-3 | 2-4a | 2-4b |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 48% | 46% | 18% | 41% | **78%** | **9%** | **67%** | **87%** | **71%** | **55%** | 20% | 53% | 13% |

Table 2: percentage of all students' success for each question of the questionnaire

The first result is the general weakness of these percentages that may be surprising because they were collected among science majors.

The second result is the relative ability of these students in computing limits of functions such as and (questions 2-1 and 2-2) by applying the algebraic rules they learnt the year before. The balance between well-answered questions and less successful ones can be explained by some mistakes or also some omissions in the memorizing of theses rules. The specific poor result in 2.1.b (9%) can be explained by the habit students have to consider only positive values of real numbers whereas the very good result in 2.2.a (87%) can be explained by the non-existence of an indeterminate situation. In the same way, the application of algebraic rules leads to relatively good results for question 2.4.a (53%).

The third result is the weakness in question 2-4-b (13%) which shows the difficulty for most students to change their mind and adopt a local view of the function near 2 and to consider the limit as the derivative number of the *ln* at 2.

The fourth result we found is that only 3 students answered both questions 1.3 **sin(2π*n*)** and 2.3 **cos(2π*x*)** correctly, revealing a clear difference between two groups of students : those whose conception of function would be mostly at the punctual level and those whose conception of function would be mostly at the global level. We decided to divide the student population according to their answer to question 1.3, that is to say according to **their possible level of conception** : 104 students answered that **sin(2π*n*)** has no limit when the integer *n* approaches ∞, revealing for these students a global vision of the function **sin**, as an oscillating function. 64 students answered that **sin(2π*n*)** has limit 0 or 1, revealing for these students a process of evaluation for certain values of the integer *n*. We chose to group students together whose responses were 0 or 1, the heart of the problem being only their ability to reach the global level of conception. With this classification, 126 students remainded whose level of conception was unknown. it could be thought that these students are able to move easily from one level to another but, it will be seen later that these students were in fact the weakest among all of them.

With this categorisation of students, distribution of the answers to the other questions are the following :

| Questions | Answers | Total (298) | Global (104) | Punctual (68) | Unknown (126) |
|---|---|---|---|---|---|
| 1-3 **sin(2π*n*)** | No limit | 35% | **100%** | ******* | ******* |
| | **0** or 1 | 23% | ******* | **100%** | ******* |

| | | | | | |
|---|---|---|---|---|---|
| | NR | 27% | ******* | ******* | **65%** |
| 2-3 **cos(2πx)** | **No limit** | 20% | **52%** | 4% (*) | 3% |
| | 0 or 1 | 21% | 15% | **73%** | **23%** |
| | NR | 30% | 20% | 10% | **49%** |
| 1-1 $(-1)^n + 1$ | **No limit** | 48% | **80%** | **49%** | 21% |
| | If odd/even | 5% | 3% | **9%** | 5% |
| | NR | 17% | 0% | 9% | **37%** |
| 1-2 $\sqrt{n} - n$ | No limit | 12% | 11% | 18% | 10% |
| | - ∞ | 46% | **65%** | **47%** | 29% |
| | NR | 23% | 9% | 12% | **41%** |
| 1-4 **cos()** | 0 or **1** | 41% | **53%** | **75%** | 33% |
| | NR | 27% | 5% | 13% | **53%** |

Table 3: percentage of students' answers to questions 1-3, 2-3, 1-1, 1-2 and 1-4 according to possible levels of conception

For simplicity, for each of the questions 1-3, 2-3, 1-1, 1-2 and 1-4, we only considered the main categories of answers (in particular, NR means « no answer »). In the first line, the numbers between brackets are the totals on which the percentage in the corresponding columns are taken.
(*) among these, only three students answered both question 1-3 and to question 2-3 correctly.

For brevity, we will denote by (G) the first group (104 students), by (P) the second group (68 students) and by (U) the third group (126 students). This categorisation is very simple and can be criticised but it provides a contrast between students which has some definite consequences for questions 2-3 and 1-1. We think that it also has some consequences for questions 1-2 and 1-4 but lesser effect can be explained by the possibility for students in that cases to apply algebraic rules such as summation (1-2) and composition of limits (1-4).

The most important correlation is observed with questions 2-3 **cos(2πx)** and 1-1 where no algebraic rules can be applied directly. First of all, there is a strong coherence between answers to question 1-3 **sin(**2π***n*)** and answers to question 2-3 **cos(2πx)**. Whereas only 20% of the whole population gave a correct answer to question 2-3, this percentage increases to 52% among the (G) group and falls to 4% and 3% in the other groups. This result is also clear in the reverse: indeed, 90% of students whose answer to question 2-3 **cos(2πx)** is correct answered that **sin(**2π***n*)** does not have any limit when the integer *n* approaches ∞, that denotes a global conception of functions and sequences. Secondly, this difference between global and punctual permits 80% of students belonging to the (G) group to answer to question 1-1 correctly , whereas this percentage falls to 49% for the (P) group. Thus the punctual level seems to be insufficient to tackle problems of limits of functions or sequences for which there are no algebraic rules to apply. At the same time, 21% of students gave 0 or 1 as an answer for the limit of **cos(2πx)** and this percentage rises to 73% among the (P) group, much more than in the (U) group (23%), which reinforces our classification. Moreover, in the (P) group, 9% of students answer that the limit of the sequence  depends on the parity of *n*, still much more than in the (U) group (5%).

These results confirm the difficulty for students belonging to the (P) group in approaching sequences and functions as global objects, for instance with a limit which is unique. Finally, punctual and global levels of conception of function appear to be good levels of reasoning in some situations as well as traps in others. Moreover, it appears that only a few students are able to move from one level to another. Indeed, with regard to group (U), it appears that the students mostly give no answer to the questions, revealing the weakness of this group and the difficulty for students of this group to give answers by building a reasoning at one of the two levels, as soon as algebraic rules cannot be applied.

The categorisation is not so important with regard to the results for questions 2-1 and 2-2. The following table shows again the relative weakness of the groups for algebraic manipulations. There is no other major difference in these results according to the categorisation we made between (G), (P) and (U); that is to say that in each line of the table, results of students from the (G) group are better than those of students from the (P) group, which are themselves better than those of students from the (U) group.

| Correct answers | Total (298) | (G) group (104) | (P) group (68) | (U) group (126) |
|---|---|---|---|---|
| Question 2.1.a | 78% | **92%** | 81% | 65% |
| Question 2.1.b | 9% | **19%** | 7% | 2% |
| Question 2.1.c | 67% | **83%** | 72% | 51% |
| Question 2.2.a | 88% | **96%** | 94% | 80% |
| Question 2.2.b | 72% | **81%** | 71% | 65% |
| Question 2.2.c | 56% | **70%** | 60% | 42% |

Table 4 : percentage of students' answers to questions 2-1-a-b-c and 2-2-a-b-c according to possible levels of conception

This classification between the three groups can also be seen through their type of "baccalaureat" (final secondary school exam) and their results in this exam. It shows that belonging to the (G) group, that is to say having reached the global level of conception, is strongly correlated to good results in the exam.

| | Total (298) | (G) group (104) | (P) group (68) | (U) group (126) |
|---|---|---|---|---|
| Bac S | 91% | 97% | 95% | 84% |
| Passable | 48% | 27% | **61%** | **56%** |
| Assez Bien | 35% | **47%** | 23% | 31% |
| Bien | 15% | **22%** | 13% | 11% |

Table 5 : percentage of students with "*Bac S" (science majors)* , mention "*passable" (pass grade)*, "*assez bien" (c grade)* and "*bien" (b grade)* according to possible levels of conception

**4) The transition between secondary school and university**

In this last section, we have tried to link what is expected from pupils at the end of upper secondary school with what is expected from them at the beginning of university. We first collected the texts concerning functions in the two last baccalaureat sessions:

> Let us consider the function f defined on the interval ]-1,+∞[ by
> $f(x)= x - \ln(1+x)/(1+x)$
> The graph (C) of f is shown on Annex 2, which will be completed and handed back at the end of the examination.
>
> **Part A : study of some properties of (C)**
>
> 1. We denote by f' the derivative function of f. Compute f'(x) for all x in the interval of definition ]-1,+ ∞[.
> 2. For all x in the interval ]-1,+ ∞[, we set $N(x)=(1+x)^2-1+\ln(1+x)$.
>    Verify that this defines a strictly increasing function on ]-1,+ ∞[. Compute N(0). Deduce the variations of f.
> 3. Let (D) be the line with equation $y = x$. Compute the coordinates of the point of intersection between (C) and the line (D).
>
> **Part B : study of a recurrent sequence defined from the function f**
>
> 1. Show that if x belongs to [0,4] then f(x) belongs to [0,4].
> 2. Let us consider the sequence $(U_n)$ given by $U_0=4$ and $U_{n+1}=f(U_n)$ for all n in N.
>    a. On the graphic from Annex 2, using the graph (C) and the line (D), set the points of (C) with abscissas $U_0$, $U_1$, $U_2$, $U_3$.
>    b. Show that $U_n$ belongs to [0,4] for all n in N.
>    c. Study the monotony of $(U_n)$.
>    d. Prove that the sequence $(U_n)$ is convergent. Let l be its limit.
>    e. Use part A to give the value of l.
>
> Baccalauréat, 15 juin 2007, Métropole

> 1. Let f be the function defined on R by $f(x)= x^2 \exp(1-x)$.
>    We denote by (C) its graph in an orthonormal frame (O,i,j) with unity 2cm.
>    a. Determine the limits of f at -∞ and +∞. What graphic consequence can be induced on (C) ?
>    b. Justify that f is derivable on R. Determine its derivative function f'.
>    c. Determine the variations of f and draw its graph (C).
>
> Baccalauréat, juin 2006, Métropole

The analysis confirms that what is required from the pupils is relatively technical and there are no adaptations of knowledge compared to standard tasks already done during the year. Pupils had to determine variations of two functions by calculating their respective derivative.
However, questions are so detailed that no initiative is left to the pupils. For instance, in the text of 2007, the useful auxiliary function N(x) is introduced by the statement. Pupils must justify that this function is increasing. One can think that they would calculate its derivative again, whereas other more qualitative methods are possible. In these two texts, functions are

algebraic objects. The tasks expected are algebraic ones : computation of global derivatives and computation of limits by application of algebraic rules.

However, as soon as students begin their studies at the university level, they are asked to make reasoning at the local level. We collected some exercises from French university belonging to the first exercise sheets of students entering university. In the following, three exercises from University of Bordeaux are given.

> En utilisant des suites, montrer que la fonction cos(1/*x*) n'admet pas de limite en 0.

> Etudiez la continuité de la fonction f définie par f(*x*) = sin(*x*)/*x* si *x* différent de 0 et f(0)=1.

> La fonction f définie par f(*x*) = *x* cos(1/*x*) si *x* différent de 0 admet elle un prolongement par continuité en 0 ?

These examples show what can be expected from students at the beginning of their course on functions: calculations of limits and problems on continuous functions which often can not be reduced to the application of algebraic rules. These exercises require reasoning at the local level of functions. As noted by Artigue (1993), analysis is the field of approximations and the algebraic methods (equalities, equivalences of expressions…) have their limits as soon as students enter university. Moreover, the mechanisms about approximations can only rise in the long run. Average students coming from secondary school are not strong enough at the punctual and global levels and moreover there is a rupture between their old practices and what is expected from them at the beginning of university.

A few months ago, the French mathematical society (SMF) issued a proposal for a common base of knowledge for teaching students at the beginning of university. This decision follows a recent reform of French universities where knowledge to be learnt was no longer proposed by the French ministry of education and could vary from one university to another. Inside this proposal, knowledge on functions is set out as the following:

> **3) Fonctions d'une variable**
> a. Théorie élémentaire : Etude locale (développements limités), étude globale (théorème des accroissements finis …), fonctions usuelles

This proposal puts forward the idea that teaching the local level before the global one is preferable. Even if this order can be explained from a mathematical point of view, it may be considered that it is not a good choice from the point of view of students' abilities.

## 5) Conclusion

Our answers to the questionnaire confirm that teaching at the secondary school has contributed to more or less banishing the local level and has also contributed to dissociating the punctual and the global levels : for instance the results confirm the ability of students in algebraic manipulations and the difficulty for students belonging to the (P) group in approaching sequences and functions as global objects. Moreover, it appears that only a few students are able to move from one level to another and even most students cannot built a reasoning on one of these two levels, as soon as algebraic rules on longer be applied. The comparison between categorisation of students and baccalaureat success also shows that results of students from the (G) group are better than those of students from the (P) group, which are themselves better than those of students from the (U) group. It reinforces in particular the idea that having one conception (punctual or global) is preferable when treating problems where no algebraic rules cannot be applied and that having reached the global level of conception is strongly correlated to good results in the exam.

Our analysis of baccalaureat texts confirms that what is requested from the pupils is relatively technical and that there are no adaptations of knowledge compared to classical tasks already done during the year. Algebraic rules generally can be applied and this type of exercises do not prepare students to what is expected from them on entering university. Lastly, our analysis of university curriculum as well as some exercise sheets confirms that the university teaching does not take sufficient responsibility for this transition from punctual and global levels towards the expected local level.

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