A GRAPHIC MILIEU TO TEACH THE CONCEPT OF FUNCTION:

WHICH FORMS OF KNOWLEDGE MAKE STUDENTS ABLE TO CONJECTURE

AND PROVE?

Isabelle BLOCH

IUFM d'Aquitaine (Institut Universitaire de Formation des Maîtres) et DAEST (Laboratoire de Didactique et Anthropologie des Enseignements Scientifiques et Techniques, Université Bordeaux 2) France

Abstract: In many countries, the first concepts of calculus (such as functions) are taught by giving examples, noticing their properties and generalizing from them in some implicit ways. Students have no means to discuss the general truth of the statements, or to examine the validity of a theorem, depending of the mathematical field. This knowledge is nonetheless the one which is demanded by teachers at University level.
Are there activities which can be organized with students at the beginning of the calculus, and which will nevertheless lead them to work about statements and validity of theorems ? I present a teaching device concerning the concept of function, which leads students to work in a graphic milieu in order to produce mathematical statements and theorems, then discuss them and test their validity. It intends to use the procedural aspect of the graphs and moreover, it provides an appropriate milieu to link intuitive and formal knowledge (the one which is required at the University to establish proofs). I will point that after an experiment, students become able to cope with functions as objects and with the nature of mathematical statements.

Keywords: functions, necessity of mathematical knowledge, proof; situations, graphic milieu, settings and representatives.

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INTRODUCTION

In many countries, the first concepts of calculus are taught apart from problems and without all the tools of formalization. Indeed the usual introduction at upper secondary school consists in giving a few examples of functions and limits, noticing their properties in order to eventually reach generalization in some implicit ways. This is supposed to be sufficient to give students a first approach of the concepts of analysis, assuming that, later, they will learn (at university level) to prove and justify the properties introduced here. But at the University, teachers often complain that students do not show the proper abilities to prove and that they use graphs and equations as if they were kinds of "labels" of a function, instead of material mediums to express concepts and tools for proving.
In this paper, I question the knowledge built by students in the common approach at upper secondary school. I will distinguish the knowledge which is useful to simply notice a property, from the one necessary to understand and even produce a proof. I will also consider the ability of transferring the first type of knowledge to the second one, without a radical break, and examine the nature of this break when it occurs: it concerns the nature of mathematics and mathematical statements, in the sense that mathematical statements are *necessary* (the statements of a mathematical theory are linked to each other and form a coherent whole), and that a theorem allows to say if a peculiar object does or does not verify a property.
My main question could be stated as follows: what type of knowledge is necessary to make students able to produce and/or prove mathematical statements about functions and to test their validity ? And what are the situations that could lead to that knowledge?
Related to this last question, this paper also describes and analyses a teaching device in which students 17-18-year old are led to produce functions in a graphic milieu, and moreover to produce mathematical statements on these functions, discussing the validity of properties, and the field of their applications.

This paper is divided into three parts:
- first, I question the usual approach in teaching the concept of function and set out the mathematical knowledge that is lacking in this organisation;
- in the second part I analyse the settings[1] in which mathematics can work about functions, and the representatives these settings allow to use, and I show how situations can be introduced to lead to the target knowledge;
- in the third part, I present a situation with a large panel of tasks for students, and explain how these tasks "compel" them to work about the properties of functions and mathematical statements in an appropriate milieu
- as a conclusion I will take some students’ works and see what kind of reasoning they were able to produce in this milieu.

I. THE USUAL ORGANISATION AND ITS EFFECTS

In the seventies, calculus used to be taught with all the rules of formalization and proofs, like the epsilon-delta definition for limits. Since it is no more the case (in France as in many countries) teachers have to try and make students perceive the objects, and the properties of these objects, by other ways.

A standard progression found in the curricula and textbooks is, for instance, the following: students are given the graph of a function, and have to set out the properties of the related function, as seen on that graph. The teacher can tell students that this function is bounded, and how it can be seen on the graph. The students are then invited to find the boundaries of the function and to write something that makes the evidence of the property appear, in terms of an inequality.

Several works show that many students who were taught this way, are not able to apprehend the right nature of mathematical objects (like functions or limits) and to produce proofs in a calculus problem: see for example Schwarz and Dreyfus (1995) , Slavit (1997), Dreyfus (1999).

As T. Dreyfus says,

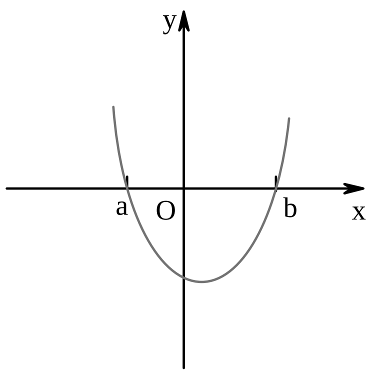
“Giving an argument or explanation is a very difficult undertaking for beginning undergraduates from at least two points of view: in most cases, they still lack the conceptual clarity to actively use the relevant concepts in a mathematical argument; and, more generally, they have had little opportunity to learn what are the characteristics of a mathematical explanation”. (Dreyfus 1999, p. 91)

The following example is a good illustration of the kind of work usually given to students at upper secondary school, and which does not give them any opportunity to learn something about explanations and mathematical proofs:

Schema 0 : bounded function

Students are given a graph, with the following instructions:

- either, “this is the graph of a function, let *m* be a lower bound – on the y axis – this is a bounded below function” (status of a definition)
- or, (a scale being given) “show that the function is bounded below” (status of a proof?)



At this time in France teaching is organised as described above:

- first, the teacher does a standard tasks in the classroom with his/her students, using different symbols of the target concept, here functions;
- then, students are supposed to do the same, with other emblematic symbols of the same concept;
- students are supposed to see in the used symbols (graphs, tables of numbers, formulae …) the same meanings as the teacher does, that is, functions and their properties.

This presentation is supposed to be more “intuitive” than a formal one; but in fact, it does not make the fundamental mathematical knowledge appear. Doing this work, students indeed cannot learn or imagine:

- which are the functions (or categories of functions) that are bounded;
- what the use of this property in the mathematical organisation is: why is it useful to study functions?
- How it is possible to distinguish this property from the others connected with order: extremes, growth…
- Which are the functions (or categories of functions) that are *not* bounded;
- What the contrary of being bounded is: the property "p" being well known, how can we enounce the property "non p"?

… And no further work will ever be done about these questions.
In other words, this ostensive way of teaching does not lead to real work on mathematical statements: it is a specificity of mathematical statements that they allow to know what properties they determine, what mathematical objects verify these properties and what are those that *do not* verify the properties. But to do that we need tools to valid a property, otherwise we cannot do anything with it. And if we know a property we can also know its contrary, which is not possible in the present organisation. Students are then going from a representative to another without knowing the use of it.
As Schwarz and Dreyfus say[2], in mathematics "learning is reduced to mapping between several notation systems signifying the same abstract object". And in the same paper the authors point the fact that research studies about learning functions and graphs show persistent difficulties in linking those different notation systems:
- Students do not succeed in tasks linking information from different settings (from a formula to a graph, even from one graph to another graph…): students' knowledge is compartmentalized;
- Students have great difficulties to construct graphs, tables, or formulae by themselves, so teachers usually make them work on given notational systems and avoid tasks of construction.

The authors insist on the ambiguity of all representatives of a function, and on the fact that teaching often does not deal with this ambiguity: rather often, algebraic ambiguities are dealt with because it is the nature of algebraic work to look if two formulae represent the same function, but the treatment of ambiguities about graphs or tables depends of other external factors as curricular goals or grade level or even the particular problem: in short it is treated in the didactical contract. For instance, depending of the context it is considered obvious that a table (two numbers and images) is a representative of a linear function, or that a curve like the one above (schema 0) is the graph of a quadratic function.
Schwarz & Dreyfus conclude that "ambiguity problems are avoided in standard curricula because students do not have the tools to cope with them". (Schwarz & Dreyfus 1995, p. 263)
At the same time Duval (1993, 1996) studies the partiality and ambiguity of representatives (every representative is partial to what it represents, and partiality leads to ambiguity) and concludes that we must consider the interactions among different representations of a mathematical object as being absolutely necessary to construct the concept. We follow Duval about this necessity, but as he says that mathematics work with and on "representations" of their concepts, we think that it is important to distinguish *representations*: the way people imagine the concept, from *representatives*: the way symbols are used to make the concept "appear". So according also to Schwarz & Dreyfus, we will call *representative* of a function a table of numbers, or a formula, a symbol "*f*", a graph…

But if “mapping between several notation systems signifying the same abstract object” is necessary, students’ difficulties will not make it easy to organize the work through various representatives of functions, in order to link the different settings. Two representatives of the same function being given, how can we be sure that for students they signify the same object, or even that for them they signify *something*? To try to answer this question, I will first explore the possibilities of representation that the different settings offer, and retain some tasks to link representatives of functions: some of these tasks are rarely given in the usual organisation of teaching, although they are interesting to ensure – as far as possible – that students work on “the same abstract object”.

The second question is to organize situations that permit to validate, that is, to work on real mathematical knowledge and mathematical statements. And speaking of functions, the mathematical knowledge is relative to the *properties* of functions, and not only to what we can “see” about functions with some representatives.

Our device thus proceeds from the same analysis as D. Slavit’s, who said he wanted to develop a *property-oriented view* of functions:

“… a property-oriented view is established through two types of experiences. First, the property-oriented view involves an ability to realize the equivalence of procedures that are performed in different notational systems. Noting that the processes of symbolically solving $f(x) = 0$ and graphically finding x-intercepts are equivalent (in the sense of finding zeroes) demonstrates this awareness. Second, students develop the ability to generalize procedures across different classes and types of functions. Here, students can relate procedures across notational systems, but they are also beginning to realize that some of these procedures have analogues in other types of functions. For example, one can find zeroes of both linear and quadratic polynomials (as well as many other types of functions), and this invariance is what makes the property apparent.” (Slavit 1997, p. 266-267).

In this extract it is also apparent that students need to do lots of comparisons between different functions in different notational systems to realize the invariance of properties; but not any task will help to reach this aim. This means that if the choice of pertinent representatives, and different classes of functions, are necessary to reach our aim, it is not sufficient: the situation in which students are immersed is essential to produce the target knowledge. By situation we mean the type of problems students are led to solve: with these problems we try to obtain a work on mathematical statements and an activity of reasoning in the classroom. The situation we implemented in a class is described in III.

II. REPRESENTATIVES, SETTINGS AND POSSIBLE TASKS

II.1 Settings and representatives

In the standard progression, graphs are used because they are seen as an easy way to *show* functions, that is, as “good” representatives of functions: good for teaching of course, to present the concept of function with a real time (and calculation) saving process. It is supposed that students can see functions and their properties through graphs. Yet, we have noticed that the usual treatment of graphs does not seem appropriate to the purchased aims. But what are the other possibilities, in the same setting or in different ones? And how can each representative open the way to the concept? What is it possible to do to link various representatives in different settings?

The settings at our disposal when working about functions are the following:
- numerical: tables of values;
- algebraic: formulae, equations;
- geometric: variable geometric magnitudes;
- graphic: straight lines, curves, axis;
- formal: notations f, f^{-1}, fog, $f(x)$, …
- analytic: with notations as ∞ , or related to orders of magnitude; it is used for heuristic but not for any validation at upper secondary school.

These settings have not the same properties for mathematical work about functions; one or another representative in some setting does not allow the same validation for the same problems and does not show the same properties of the function. It is important, first to know the properties of each setting, in terms of how they are partial to the objects they represent; and then to make an inventory of the tasks they allow to organize for students. Teaching depends also on the former knowledge built by students in each setting.

- In the graphic setting

One can see only what is in the window, one cannot see “as far as” the infinite limits; the graph is discrete (and so are graphs on a graphic calculator too) so one cannot “see” the continuity, it must be supposed. But the curve can be seen as an object, which is interesting so as to see properties; on the other hand one also can see *false* properties, e.g. if a function has got the limit zero at infinite, its derivative has got the same limit zero… Therefore, this setting cannot be used *alone* to validate and prove, this is why we must introduce tools of another setting to *operate* on functions. Otherwise, students do not possess important former knowledge about graphs.

- In the algebraic setting

One can see what type of function one has got, e.g. polynomial, and then deduce well-known properties; one can transform the formula; but one cannot see the curve, neither values or roots. This setting can be used to prove, but it soon becomes very difficult if functions are more than linear or quadratic polynomial ones. Moreover we know that students at the present time do not do very well in algebra, so algebra is a setting where they have only got little knowledge; and teachers encounter great difficulties when they try to let them work on formulae to prove growth, or limits and derivatives. So the algebraic setting is useful to help students to prove, but it is not so good for intuition, and it reveals its limits as soon as functions get rather complicated.

- In the geometric setting

The geometric setting is not much used at the present time: problems of variation of a geometric magnitude are no longer the object of an important work, as they could have been in the 50-60ties. So students are not used to working on this kind of problems, and cannot be credited of the linked knowledge. It would be very expensive, in terms of teaching strategy, to reintroduce this setting in the classroom work.

- In the numerical setting

This setting inherits a lot of students’ knowledge, but it is very partial to the represented objects, and therefore it carries lots of ambiguities: on a numerical table you can see only a few values, and you could infer that the function is linear, or has got an extreme, even if this is not true. Besides, this setting is not convenient to prove, because it is discrete whereas all functions are continue at this level; and the continuity cannot be induced on a numerical table as on a graph. Anyway this setting is useful to help to draw graphs, that is the way it is used in the usual organisation: at best as an abacus, which is a local view of functions (point by point) and not a global one (the mathematical object "function").

- In the formal setting

Students hardly know the formal setting, and it is the best one for validation, the one which will get the greater place at University. One could imagine that one of the objectives at the entry into calculus is to try and familiarize students with the work in this setting. But this formal setting cannot be sufficient to give meaning to the concepts, so it must be coupled with another setting where students can be confronted with formulae or graphs.

We can notice that the graphic setting and the formal one seem interesting, but the algebraic one could be of some help; and anyway, after having recognized the useful settings, we must now make an inventory of the possible tasks, and choose the ones which obey to at least two conditions:

- first, being adequate to make students construct functions and work on their properties, as to ensure that students work on mathematical knowledge as far as possible; and this involves possibilities of operating on functions;
- secondly, being rather easy to introduce in the classroom work: this ergonomic condition is essential to give such a device a chance of being achieved.

II.2 A choice of tasks

We will not repeat the well-known tasks such as study the variations of a function when it is given as a formula, calculate a few values, and draw the graph. One can see that there are lots of interesting tasks that are not the object of a work at upper secondary school. Only in the graphic setting can we mention:

- find images and antecedents to validate properties;
- enlarge the window or make a zoom;
- change the axis and discriminate between what is preserved and what is not;
- change the scale, see that the concavity does not change …[3]

But this is the conversion between settings that provides us with lots of non usual tasks, for instance:

- find information about the properties of a (class of) functions on a graph;
- find the algebraic formula of a function knowing the graph, values, and the type of the function (linear, or polynomial, or square root…);
- construct graphs under conditions (bounded function, given roots …)
- see how a graph or an equation can be the one of a composite function;
- compose graphs to find the graph of the composite function;
- find the graph of the inverse function;
- operate on functions through their graphs (make the structure of algebra of functions appear);
- write operations on functions, inverse, composite functions, with formal symbols and prove the graphic conjectures …

We notice that the graphic setting provides us with lots of interesting tasks, but as we said, it cannot be used alone to prove: it is useful to conjecture, and coupled with a convenient formal tool (see III) it allows a large choice of functional problems.

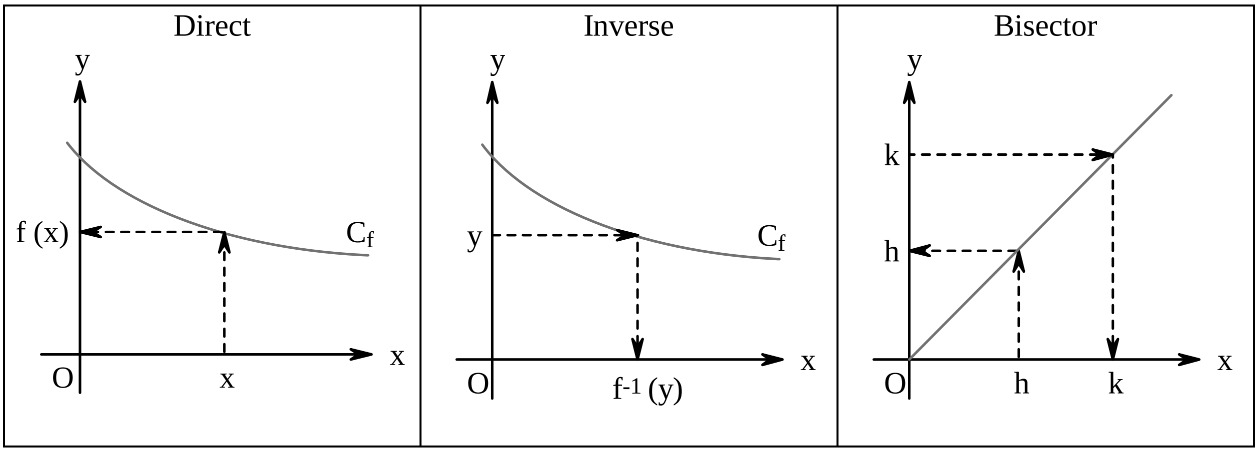
The question now is, how to organize such a work at high school level, or at the beginning of University, when students do not have theoretical means of proof at their disposal? And how can

we link a given work about graphs or equations, and mathematical knowledge, as D. Slavit recommends it?
I have used the Theory of situations, due to Guy Brousseau (Brousseau 1997), to build a teaching device for 17-year-old science students within a graphic milieu; this milieu is proper to make students draw graphs of functions and study the functions' properties, to produce and question statements about functions. In order to achieve this aim, I have borrowed the graphic milieu proposed by Pedro Alson (Alson 1989), and I organized this milieu by didactical variables that I could identify and handle in order to produce questions of analysis. I could then observe the knowledge that was expressed by students while they were working, and try to link their knowledge to theoretical knowledge. The main hypothesis that led to conceive this device, is that semiotic tools largely determine the kind of mathematical work that can be done. This hypothesis asserts that mathematical knowledge is strongly linked with semiosis, and even more, semiotic tools both *contain* and *produce* some kind of mathematical knowledge. We will now see examples of that knowledge.

III. THE TEACHING SITUATION

Pedro Alson teaches mathematics at the Central University of Venezuela, in Caracas. At their entry at the University, his students had very poor abilities in algebra, so he tried to find another way to make them perceive the concept of function; and more than perceive, *work* on functions without formulae. When I borrowed his device, I had to organize it according to the French curriculum of a scientific course for 17-18-year-old students, and to plan right phases of the situation.
In this situation, students have graphs at their disposal, and they must build graphs of functions, with some constraints expressed in the instructions, like a fixed value or an inequality required to be verified on an interval. I shall express graphs of functions by CGR (Cartesian Graph Representation). Students must also justify that the CGR – either given CGR or CGR they draw - is consistent as the representative of a function, and that the properties of the graph are coherent with those of the function.
To do these tasks, students can use a *path*. A direct path starts at a point S (x, 0), follows the directions of the axis (first the y one, then the x one) and goes through the corner C (x, f(x)) to the end point: E (0, f(x)). An inverse path starts from the y-axis. The paths can be used: 1) to justify that a curve represents a function, 2) to build new curves from others (for example sum or product or composite of two functions, or inverse of a function). When paths are not sufficient to prove a property, an algebraic work is required (see examples below).
Formal notations are introduced to operate with paths, and they become a help and a necessity, so that the formal work is an important part of the situation but is not introduced in an arbitrary way. In that way we obtain a *formal / graphic milieu* that is convenient to the mathematical work required. In the whole situation, students need tools to prove that a graph is actually the representative of a function: paths will be needed to do it. The rule is as follows: a graph represents a function if there is **only one direct path** from each point of the x - axis (assuming that x belongs to the domain of f) to the y- axis, with a corner on the graph of f.
There exist three basic paths:

Schema 1



Paths are given by the teacher in the first phase of the situation, as tools to make sure that a graph is a representative of a function, and to find an image or an antecedent through the function.

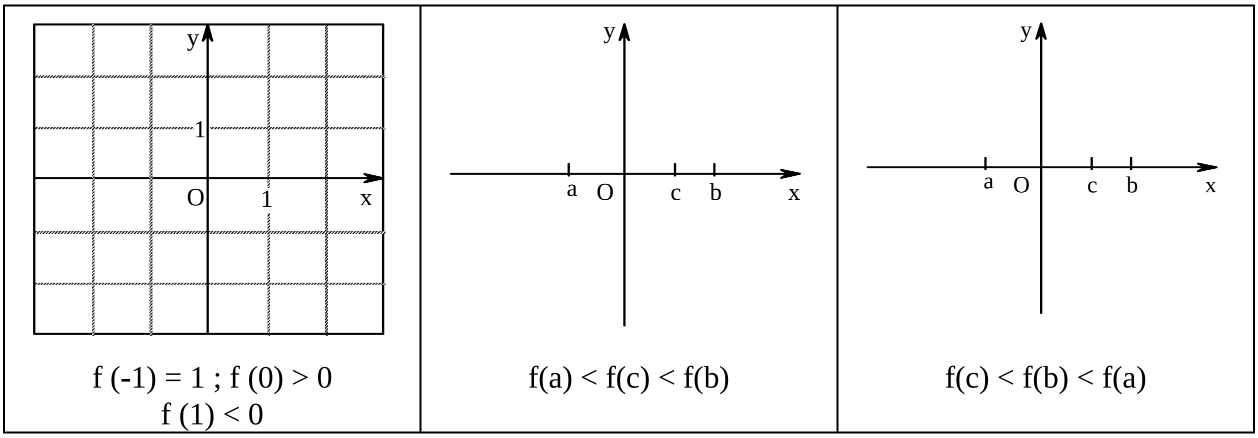
III.1 First phase

The aim of this phase is to let students produce graphs of functions, and interpret them in relation to well known properties of functions (like being increasing, or having fixed values on fixed points). This first phase leads to the first family of situations: situations of graphs production.

Example:

« Draw the graph of a function f satisfying the given conditions »

Schema 2



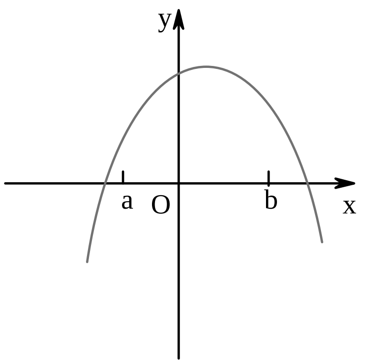
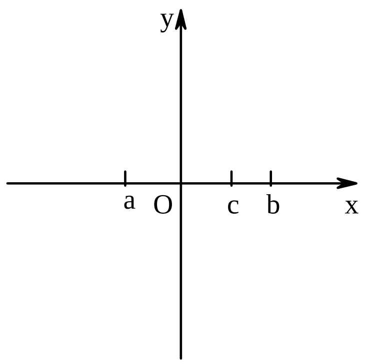
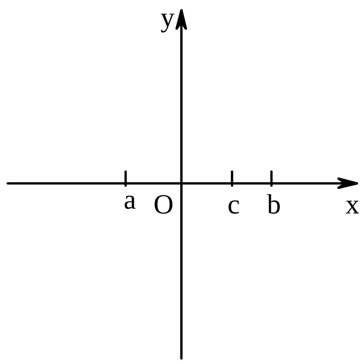
The experiment shows that students are not used to drawing graphs of « arbitrary » functions, and that they need some time to get used to doing it. It is also necessary that students should get used to paths, and convince themselves that paths are easy tools to conjecture and prove about functions. Observations prove that this phase is quite useful, since as students find it difficult to become familiar with all sorts of functional graphs, and to dare draw a graph, and be sure it is the one of the required function.

In this phase, paths are not a better tool than the "horizontal test line", and cannot be credited with creating very interesting knowledge about functions. That is why we need to organize other phases to lead to real mathematical knowledge. The first phase can be considered as settling the first level of milieu, what we call the “objective milieu” in the theory of situations, because it supplies students with pertinent “objects” to do the required work, such as various graphs, paths, and rules to use the objects. The objective milieu is a basic one since it is essential to provide students with procedural knowledge about functions, that is, means to do the further work. Students anyway also know previous objects, like linear functions and use of tables: the device will take this previous knowledge into account. We also notice that during this phase, *lots* of graphs and reasoning on graphs are useful to establish this basic milieu.

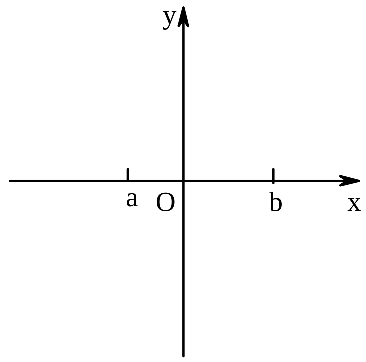
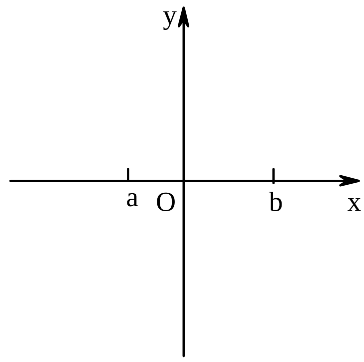
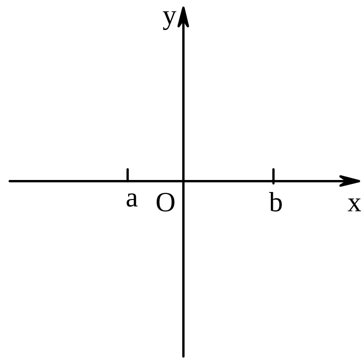
III. 2 Second phase

This first group of graphs (schema 3) leads students to draw particular graphs with particular values a, b, c; the next one must compel them to introduce **quantifiers**, and to see the difference between a punctual condition, such as f(a) > 0 for a given value a, and a quantified condition such as below, which introduces the necessity of intervals.

Schema 3

		
Tell if f(a) < f(b) or f(a)>f(b)	Draw the graph of f, such as f(a) < 0 ; f(b) < 0 ; f(c) > 0	Draw the graph of f, such as f(a) > 0 ; f(b) < 0 ; f(c) > 0

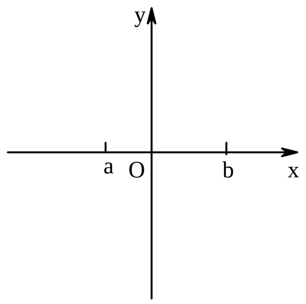
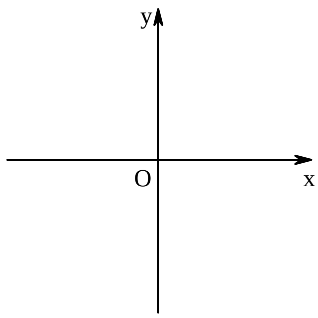
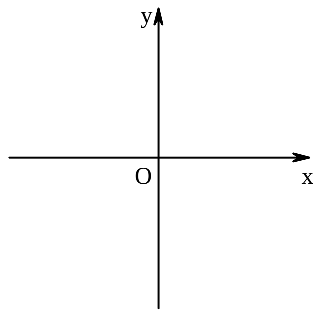
Schema 4:

		
$\forall x$ / $x < a$ or $x > b$, $f(x) < f(b)$ and $\forall x$ / $a < x < b$, $f(a) > f(x) > f(b)$	$\forall x$ / $a < x < b$, $f(a) > f(x) > f(b)$	$\forall x$ / $x > b$, $f(a) > f(x) > f(b)$ $\forall x$ / $x < b$, $f(x) > f(b)$

In the first case of schema 4, a function satisfying the required conditions **cannot** be **continuous in a**. As students are at the beginning of their course of calculus, they do not know that a function may be discontinuous, therefore the question whether a function can do a « jump » like this is very likely to be discussed; the paths allow to answer the question, because a discontinuous function does verify the good condition with a direct path applied at the right and the left of a. So we see that the work on given conditions can lead to **unknown functions** for students. Therefore the given conditions are *didactical variables*[4] of the situations. By acting on these conditions we can introduce the students with new functions.

III. 3 Studying properties of functions

The following phase consists of studying properties of functions which are linked with the order on R, such as to be bounded, or increasing or decreasing. This leads students to make the difference between a condition like: « f is bounded by f(a) and f(b) », which is expressed with one universal quantifier:
$\forall x \in [a, b]$, $f(a) < f(x) < f(b)$, and a condition like: « f is increasing on [a, b] » , which requires **two** universal quantifiers: $\forall x \in [a, b]$, $\forall y \in [a, b]$, if $x < y$, then $f(x) < f(y)$. There is a discussion between students, some of them being sure that the first condition means the same as the second one, that is, that a bounded function (like the one in the first space below) is necessarily increasing. A counter-example has to be produced by some students, as a curve which satisfies the condition but is not increasing. It's a work about the **meaning of quantifiers.**
Schema 5

		
f is bounded on [a, b] by f(a) and f(b) and f(a) < f(b)	On **R**, f is bounded above but not below	On]-2, 1[f is not bounded above

The condition of the last case in schema 5 is quite difficult, since students at this level do not know curves with asymptotes – and even less more pathologic functions. They are trying to express the condition with quantifiers. First, they try ($\forall M \in$ **R**, $\forall$ x $\in$]-2, 1[, f(x)> M) but they soon discover that such a function would be hard to imagine: they cannot draw any graph with this condition. So they find that a better condition is ($\forall M \in$ **R**, $\exists$ x $\in$]-2, 1[, f(x)> M) and try to find a convenient function, but it is not easy in a graphic milieu, because it is not possible to build a no bounded function that can be seen in a convincing way (the window is limited); the teacher has to give them a function with a formula, $f(x) = 1/x^2$, so that they can verify with algebraic calculation that it is not bounded; moreover verifying this is not easy for them. It shows that in some cases, the graphic work must be articulated with the algebraic one to enhance the understanding.

When trying to prove that the function $f(x) = 1/x^2$ is not bounded in the interval [0,1] students use different methods: some of them try to show that it can exceed 60, because for them “60 is a big number”; others show that it exceeds 10^{98} because it is nearly the biggest number of their calculator; and a few ones say: “Let us do it for every M, such as to be certain”. When they write the solution, $\forall x$, $x < 1/\sqrt{M}$, then f(x) > M, they have a problem because $x \in [0,1]$, which is not sure for every x if M <1 and $x < 1/\sqrt{M}$; and instead of choosing some $x \in [0,1] \cap [0, 1/\sqrt{M}]$ – the intersection is not empty – they say: if M<1 the property is trivial, so let us do it for M>1 only, and we are sure that this way $1/\sqrt{M} < 1$ and we stay in the interval [0,1].

We can identify this reasoning as typical of analysis, it is a reasoning by a sufficient condition, even if it is not the expected sufficient condition (find *one x* in the intersection is sufficient and always possible). So in this case the graph helps to conjecture that it could exist no bounded functions in a bounded interval, but the convincing proof takes place in the algebraic and analytic settings.

Of course there are three very different levels of “proof” in this work:

- the first one, the proof that there exists an *x* so that *f(x)* > 60, has the status of a calculation, no more;
- in the second one, 10^{98} appears as a “generic number” to convince that *f* can exceed a “very big” number, so it is not bounded;
- the third one is a real analytic proof.

So we can see that this phase is the key point of the device, because it leads to the aimed knowledge – analytic thinking – and makes students work on the properties of functions: what are

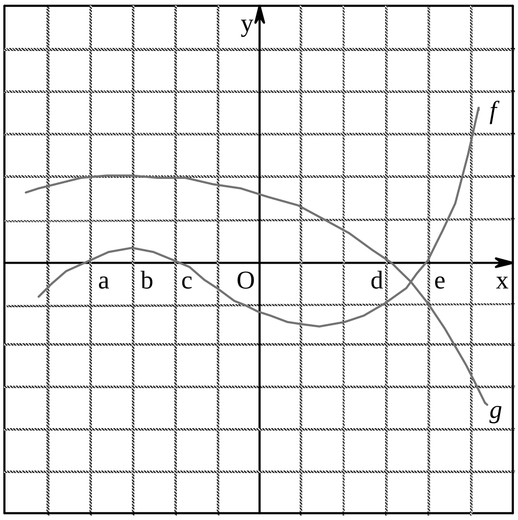
the functions that verify a property, what is the meaning of a property such as to be bounded, or *not* bounded. It also leads them to use the formal setting, such as quantifiers, and to discuss their number and place; then it leads students to complete analytic reasoning by sufficient condition.
The milieu of the third phase is thus a part of the *learning* situation, the one that permits to validate and argument.

III.4 Drawing graphs for sums and products of functions

This fourth phase must lead students:
- to enhance their comprehension of how to obtain new functions and study them;
- to consolidate their knowledge about numbers and particularly the effects of the product of two numbers. It seems trivial, but when you give the following tasks to 17-year-old students, you get surprises: at first, students are not able to see that the product of a negative value of f and a positive value of g gives a negative value, or that at a point where $f(x) = 1$, then $f(x)\times g(x) = g(x)$;
- to strengthen the use of paths, and to make students able to anticipate the result of a programmed operation;
- to express general rules (on the nature of the curve of the inverse function when it exists, for example, and the conditions of its existence) and to use their knowledge about affine or quadratic functions, by solving problems on the product or the sum;
- to use the formal setting much more than they have done in the first phases, and experiment its efficiency for anticipating and proving;
- to link graphic, algebraic and formal settings.

Examples of instructions: schema 6 *(The instructions are given for the product, but the same work can be done with the sum.)*

1) Draw the graph of the product function of f and g, using the values a, b, c, d, e.



2) What rules can you express about the graph of $f \times g$ from the graphs of f and g? Express rules about:
- the intersection points of the graphs of f and g with the x-axis, and the intersection points of $f\times g$ with the x-axis;

- the sign of $f \times g$ and the signs of f and g;
- the values of $f \times g$ and the values of f and g (for example if $|f(x)| < 1$, $|(f \times g)(x)| < |g(x)|$, and this can be seen on the graph; if $f(x) = 1$, then $(f \times g)(x) = g(x)$, one sees an intersection point with the graph of g).

3) What could you say about the product of a constant function and an affine function ? of two affine functions ?

What kind of work is expected on these questions? It is first related to students' knowledge about numbers: sums and products of numbers, and their properties. And it is much easier for the sum than for the product.

a) Sum of functions

The work on the sum could be realized without the square pattern on the graph, with a compass; but it is faster with the values, even if these values are approached ones. Students have no doubt that the sum of two linear functions is linear; it is a bit more difficult to deduce the sign of the sum, from both signs of f and g: it is necessary to think of their absolute value for each x, which is a knowledge not easy to reach at secondary school. A very clever manner to do the sum is to take the point $(x, \frac{1}{2} (f(x) + g(x)))$ which is the middle of the ordinates, and take its double. Students find that way, but it is better to let them work with the values, because the compass does the work "alone", and avoids thinking about numbers.

b) Product of functions

Students have to draw many graphs of products to get used to this way of finding new functions. First, they say that as for the sum, the product of two linear functions is linear; but as it is easy to draw three points of the graph, and saw that it is false, they become very careful: they dare not say that the product of a constant function and a linear one is linear, although the points seem to be right well in line! It has to be proved with the calculation of $c \times (ax + b)$. In the case of two linear functions, as it leads to quadratic ones, it is possible to link the characteristics of the factors and those of the product in an interesting way; this work concerns the "mapping between several notation systems signifying the same abstract object", here the graphic setting and the algebraic one. It must be pointed by the teacher that the algebraic setting is a mean of proof when the graphic alone cannot do it.

With all these interactions between the numeric, algebraic and graphic settings, this work on the algebra of functions is useful to institute a new objective milieu (creating new types of functions); but we now want to make students work about different kinds of proofs, that is, in a new situation: this new situation must be adequate to mathematical discussion, and does not only possess heuristic virtues; it is a *learning* situation (as Brousseau defines it, a learning situation is a situation where mathematical knowledge is validated and institutionnalized).

This situation must be organized in order to "compel" students to use the constituted objective milieu for proving, so this new situation is "inversed" in respect to the previous one: we call it the *dual* situation. In this new situation, the instructions change: the new goal is to find the previous given variables. The sum or the product is given with some conditions, and student have to find the factors in order that the product verifies the given conditions.

Examples:

- How could you take two affine functions the product of which would be a quadratic function with the summit of the curve at a fixed point ? With roots fixed ? Could you obtain a quadratic function without roots?
- How could you obtain a polynomial function of degree 3? What are the ways you can obtain an increasing function?

The first milieu plays as a help and a resource to do the work, because students have experienced the product of lots of functions; but they have to construct the right knowledge to see that the factors *cannot be* different from the expected one … , and to prove that it is so. And this is what we call the *necessity* of mathematical statements; students experience it with this work. Necessity is opposed to *contingence*: in the traditional way of teaching, knowledge is introduced by the teacher as a contingent fact ("Let us take a graph like this…", "When I do this product I see that…"), but lots of students are not aware of the constraints of the compatibility and the coherence of mathematics: we have met students who thought that $(a + b)^2$ could be equal to $a^2 + b^2$ if mathematicians decided it! In the same way the product of two linear functions could be linear… and they discover that if you want a quadratic function as a (non trivial) product you cannot help but take two linear ones, and not *any* linear ones if there are specificities of the result; and that doing a product of first degree terms there are functions that you can never obtain, such as a quadratic function without roots… For some students this is quite new; and for all of them it leads to discuss why they are *sure* of their result, that is, discuss mathematical truth.

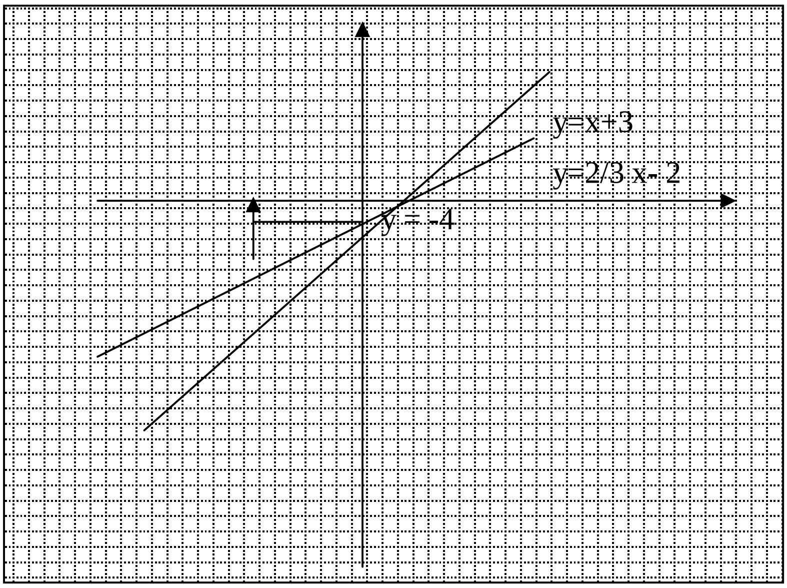
III.5 The graphic milieu and the incorporated knowledge

At the end of II, we said that semiotic tools contain mathematical knowledge that reveals with an appropriate task. We have already seen that some knowledge about numbers and the nature of functions comes from a task as find the product of two functions in a graphic milieu, or imagine two functions the product of which is given. In this part, I want to present another example of this knowledge. It is well known that if students have an equation such as $a \times b = 0$ to solve, they say that $a = 0$ or $b = 0$; and if it is an inequality $a \times b > 0$, they say "$a > 0$ or $b > 0$", which drives their teachers to despair.

In the graphic setting it is possible to link an inequality to the equation of the corresponding straight line, and to solve inequalities: for example $x + 3 < 0$ can be seen graphically as the values of the x-axis where the ordinates of the points of the line are negative. You can use inverse paths to visualize them (one path is drawn on the graph): all inverse paths for $y < 0$ cut the line at values such as $x < -3$.

Let us take two of these inequalities, the second one being relative to the sign of $2/3\, x - 2$. Then with the straight lines you see that for the values in the interval $]-\infty, -3[$ both $x + 3$ and $2/3\, x - 2$ are negative, for the values in the interval $]-3, 2[$ $x + 3$ is positive and $2/3\, x - 2$ is negative, and that for the values in the interval $]2, +\infty[$ then both $x + 3$ and $2/3\, x - 2$ are positive.

Schema 7



It allows to solve the *product* inequality: $(x + 3)(2/3\ x - 2) > 0$, but it shows something more, and the fact that you can see this property is due to the graphic semiosis you are using. For that type of inequality ($a \times b > 0$), you could think that you encounter four possibilities: a > 0 and b > 0, a < 0 and b < 0, a > 0 and b < 0, a < 0 and b > 0.

But in a graphic milieu you realize that one case never happens with lines, there are only three cases and not four, because you cannot get $2/3\ x - 2 > 0$ when $x + 3 < 0$. Of course this can be rediscovered doing a table of signs for the product $(x + 3)(2/3\ x - 2)$, but the aim of this work is precisely to do understand why the table of signs only shows three possibilities. In the algebraic setting indeed you can notice the same fact but you cannot explain it, whereas in the graphic setting you see that, given the two lines, it cannot be *otherwise*: when one line exceeds the other one, it cannot go back underneath.

This example shows how we can say that each setting of semiotic tools carries some specific mathematical knowledge; and this allows us to say that one representative or one setting does a part of the work itself. The question is then: how to explain what you have found, and the explanation is a part of the mathematical debate.

III. 6 Inverse and composite of functions

To deepen the formal control that students can gain by working within this device, it is interesting to go on with a work on inverse and composite of functions; the other aim is to make them work on functions as objects, as it is clear that both the questions and the notations concern a global view of functions. First the students have to built the target function with a heuristic method, by drawing points, and this work must be done with paths, especially the bisector path; but very soon the questions focus on the existence of the inverse, or its properties, which both involve the function as an object.

1) For the inverse:

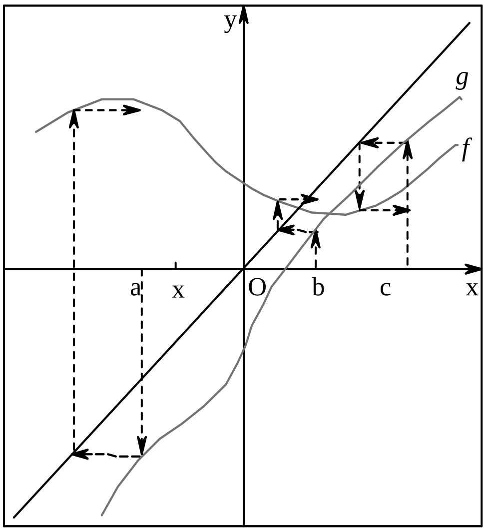
- using the paths, build the inverse of the function *f*, that is, the function that sends *x* to the antecedent of *x* by *f*. It is easy if you give a name to all the "corners" of the path: you take x and send it by a vertical path to the point *(x, x)* by the first bisector, then draw a horizontal path to the curve and obtain $(f^{-1}(x), x)$ because the ordinate is *x* and it is a point of the curve; a vertical path to the bisector gives: $(f^{-1}(x), f^{-1}(x))$ and a last horizontal path gives $(x, f^{-1}(x))$ which is a point of the graph of the reciprocal function, when this function exists. But it is not so interesting to construct the curve point by point, as to try to find the properties of functions having an inverse, and investigate and decide how to find the whole CGR.

The questions are:

- What type of functions is likely to get an inverse ? How could you quickly draw the graph of the inverse, knowing the one of the function ? Express a rule and prove it. The graphic milieu makes the answer easy because with the paths you immediately see that the second path, the horizontal one, has only one intersection with the curve if, and only if, the function is strictly monotonic. And otherwise you get two antecedents, which means two different possible inverses; so students are led to divide the domain of f into intervals where it is monotonic.
- build the graph of the inverse of the square function, on any interval wherever possible; what do you find? Can you express the two functions you have found with a well known function?

2) For the composites of functions:

Schema 8



- using the paths, build the composite of the two functions *f* and *g : fog* . Point by point it is the same method as for the inverse, take x, by a vertical path obtain *(x, f(x))*, then by the bisector *(f(x), f(x))*, then *(f(x), g(f(x)))*, and a last horizontal path to *(x, g(f(x)))* which is the point you want to reach. When you go horizontally with a path you change x, and when you go vertically you change y. A point on the bisector can be recognized because it has abscissa equal to ordinate.
- is *fog* the same as *gof* ?

Students become able to express rules about the functions that get an inverse, and the property of the CGR of f^{-1} being symmetric to the graph of *f* with relation to the first bisector; they can draw composites very quickly, and anticipate their properties. The interest of this last phase is mainly to lead students to use formal notations much more than they usually do, and to experiment the efficiency of this setting to anticipate properties of functions and to prove these properties. And moreover, in this phase the work emphasizes on functions as objects – global point of view – when students try to find the inverse of a function, and theorems about functions that get an inverse.

IV. CONCLUSION

The aim of this paper was to prove how it is possible to construct an interesting situation with a theoretical control; anyway it is important of course that the situation encounters some success.

A whole class of 17-year-old science students followed the device in 1998 (35 students). At the end of the year, I gave a questionnaire to four classes of such students, including the experimental one (140 answers). Three questions of the test were issued from Schwarz and Dreyfus paper (Schwarz and Dreyfus 1995), and wanted to test the skill about the ambiguity of representatives in the functions field: given some aligned points of a graph, could it be the graph of a single affine function for example, or of any arbitrary function whose CGR contains the given points? Or, given a CGR with asymptotes, and four different formulae, what could the equation of the function be? And another question was to build a new function (actually, a graph) with three "pieces" of graphs, if it was possible to put them together to do a new graph, and express the function with one formula or more.

Three other questions tested the common contract about functions, and the given tasks were quite traditional at that level, such as read the roots of a function on its CGR, or describe the variations of a given function.

Three questions were borrowed from a questionnaire given by Funrighetti & Somaglia (1994), to test links between the algebraic and graphical points of view, and these questions were rather classical too, such as see the growth on a graph. (I will not detail them).

At last, three questions were built to test the students' knowledge on limits and derivative from a graphical point of view, since the test was given after a whole course of analysis. These are not very classical: they must recognize the graph of the derivative of a given function (the graph of the function being given); they must compare the derivatives of two functions, knowing that their graphs are quite the same with consideration of a translation; they must determine if a graphically given sequence could have a given limit.

Then there were six classical questions and six non classical ones (where ambiguity, functions as objects, or links between settings took place).

We read the papers of students trying to analyse their procedures[5]. The results of the questionnaire indicate that knowledge about arbitrary functions, links between settings and CGR have been rather well understood and taken in, in the experimental class.

In a more general way, students of the experimental sample indicate more irregular success than the others, but they tried to solve more questions. It seems that they manifest a greater dispersion of their knowing, which is also confirmed by the fact that in the non experimental classes we could distinguish between the papers of students who usually succeeded well in mathematics from the papers of the other ones, while this was not pertinent for the experimental class because all students tried a heuristic work: they were trying to answer the questions with various means and made some mistakes, but none of them was helpless and unable to answer.

Moreover:
1) We observed that the students of the experimental sample expressed much more mathematical knowledge, and were able to discuss the relevant knowledge with pertinent proofs: discuss if a graph "could be" the graph of a given function, or if the graph of a derivative "could look the same" for two functions; and argue that a given function – a formula – "must have" some different asymptotes so it could not be the given graph. They were able to discuss with direct proofs and constructed graphic counter examples, whereas the other students could only answer if there had been a similar case in their previous experience (I can answer because it is exactly like …).
2) The answers to the questionnaire show that the students of the experimental sample had more various and relevant ways of investigation for problems and more initiative; for instance they tried to draw another graph, to calculate some values, that is, they were able to use their knowledge in a decontextualized way and to adapt it to new mathematical objects.
However very good students of the whole sample can be credited of the same abilities, and the difference comes essentially from the average ones. So I can say that good students gain the wanted knowledge without a special training, and they succeed whatever the teaching device may be. This is a well-known result anyway, and we are not surprised to see it confirmed.
3) The manifested knowledge is related to the one which is required at university level, that is, produce mathematical statements, discuss their validity, and be able to prove them; and moreover, students' abilities on graphs are those who are required at college and university for advanced students: that is, use the graphs as an heuristic way of research (Bloch 2000, Maschietto 2001). This result shows that it is possible (and desirable, considering that it will be needed later) to broaden the role of the graphic setting in the solving problem activity at secondary level.

In that way, I can say that the experimental device is more appropriate than the classical ones to further studies in mathematics, because it makes students able to produce functions, to discuss their properties, and to express and prove theorems; it also leads them to use the formal setting, which will prove very useful for their further studies. Even good students may benefit from such mathematical work.
This result is all the more interesting since a lot of French students fail in their university studies because they do not understand the game mathematics want to play at that level: ask conjectures, get various elements in different settings in order to investigate, and produce a formal proof.

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[1] For using « settings » instead of « representations » see Schwarz & Dreyfus, 1995.

[2] Schwarz & Dreyfus, 1995.

[3] For a more complete study, see Bloch 2000.

[4] Let us recall that a didactical variable of a situation is a parameter of the situation, whose value can be fixed by the teacher, and is likely to determine the work of the student: orient or forbid a strategy, put calculation means at student's disposal or not, etc…

[5] The complete study is to be found in Bloch, 2000.